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We present general solutions to the equations of motion for a superconducting relativistic chiral string that satisfy the unit magnitude constraint in terms of products of rotations. From this result we show how to construct a general family of odd harmonic superconducting chiral loops. We further generalise the product of rotations to an arbitrary number of dimensions.

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I. PRELIMINARIES

Particle physics models where symmetry breaking is involved predict, in many cases, the existence of topological defects, which are formed when the topology of the vacuum manifold of the low energy theory is non-trivial [1]. Cosmic strings, in particular, are line-like objects that are formed when the the vacuum manifold contains unshrinkable loops. For a review see [2].

In [3] it was shown that cosmic strings can be superconducting. In the case when the charge carriers on the string are not coupled to a gauge field the action for the string and the current can be taken to be

$$S = \int d^2\xi\sqrt{-\gamma}(-\mu + \frac{1}{2}\gamma^{ab}\phi_{,a}\phi_{,b}) \tag{1}$$

where μ is the mass per unit length of the string, γ^{ab} is the induced metric on the string worldsheet and ϕ is the field of the charge carriers living on the string. These strings were shown in [4] and [5] to have solutions in the case when $\gamma^{ab}\phi_{,a}\phi_{,b} = 0$ of the form

$$\mathbf{x} = \frac{1}{2}[\mathbf{a}(u) + \mathbf{b}(v)] \tag{2}$$

for the string position and

$$\phi = \frac{1}{2}f(v) \tag{3}$$

for the field living on the string with the constraints

$$\mathbf{a}'^2 = 1 \tag{4}$$

and

$$\mathbf{b}'^2 + f'^2 = 1 \tag{5}$$

where $u = \sigma - \tau$ and $v = \sigma + \tau$ and σ and τ are space-like and time-like parameters respectively that parametrise the string world-sheet. These strings are called chiral because the current only moves in one direction on the string.

Comparing this to the usual Nambu-Goto case one can see that $f(v)$ acts like a fourth component of the three-vector $\mathbf{b}(v)$, making chiral superconducting strings behave like Nambu-Goto ones with chiral excitations in an extra fifth dimension. Indeed, this property was used in [6] in an investigation of the properties of superconducting cosmic string cusps.

The right- and left-moving excitations, $\mathbf{a}'$ and $\mathbf{b}'$, on a regular Nambu-Goto string in Minkowski space-time are arbitrary functions that satisfy the unit magnitude constraint, $|\mathbf{a}'| = |\mathbf{b}'| = 1$. Expressions for these functions are often given as Fourier sums and the unit magnitude constraint generally gives a non-linear set of equations involving the vector coefficients of the Fourier expansion. As a result, parametrising strings beyond the first few harmonics proves to be a difficult task. Fortunately, in that case, there exists a method to generate strings involving products of rotation matrices [7] that act on a starting unit vector so that the unit magnitude constraint is satisfied trivially.

In a recent study of the properties of chiral cosmic strings [8] it was assumed that the current is constant. The work in [9] assumed that the current takes a very simple non-constant form. As was pointed out in the latter work one could expect to have loops with varying currents if the loops are formed by intersections involving different strings or if different segments of the loop or string were at some point in causally disconnected regions. For long strings this is always the case and we therefore expect varying chiral currents to be generic.

The purpose of this work is to generalise the work in [7] for generating three dimensional unit vectors to four dimensional ones that include the current as a fourth component of $\mathbf{b}(\sigma + \tau)$. We start by casting the method somewhat differently, generalising it to four dimensions and further generalising it to an arbitrary number of dimensions.

In Section 2 we show how to construct an arbitrary N harmonic unit vector in four and three dimensions. In Section 3 we use these results to construct arbitrary chiral current carrying strings that exclude overall spatial orientation freedom. In Section 4 we show how to solve the center of mass constraint and we present a family of odd harmonic chiral loops. In Section 5 we generalise the arguments in Section 2 to an arbitrary number of dimensions and we conclude in Section 6.

II. SOLUTION TO THE UNIT MAGNITUDE CONSTRAINT IN TERMS OF PRODUCTS OF ROTATIONS

We can think of the Euclidean 4-vector

$$\tilde{\mathbf{b}}' = \begin{pmatrix} b'_w \\ b'_x \\ b'_y \\ b'_z \end{pmatrix} \tag{6}$$

as having unit magnitude according to (5) with $b_w = f$. Consider the vector $\tilde{\mathbf{b}}'_N$, that can be constructed from a finite sum of Fourier components,

$$\tilde{\mathbf{b}}'_N(v) = \mathbf{Z} + \sum_{n=1}^{N} \{\mathbf{A}_n \cos nv + \mathbf{B}_n \sin nv\}. \tag{7}$$

The Fourier coefficients satisfy the set of $4N+1$ non-linear relations derived in [7]

$$\sum_{n=m-N}^{N} (\alpha_n \cdot \alpha_{m-n} - \beta_n \cdot \beta_{m-n}) = 4\delta_{m0} \tag{8}$$

with $m = 0, 1, ..., 2N$,

$$\sum_{n=m-N}^{N} (\alpha_n \cdot \beta_{m-n} - \beta_n \cdot \alpha_{m-n}) = 0 \tag{9}$$

with $m = 1, ..., 2N$,

$$\alpha_n = \alpha_{-n} = \mathbf{A}_n,\ \beta_n = -\beta_{-n} = \mathbf{B}_n, \quad n \neq 0, \tag{10}$$

and

$$\alpha_0 = 2\mathbf{Z},\ \beta_0 = 0. \tag{11}$$

These equations can be obtained from the constraint equation (5). The total number of degrees of freedom in the coefficients in (7) is $8N + 4$ so the remaining number of degrees of freedom after satisfying the constraint is $4N + 3$. Below we show how to construct (7) from a product of rotation matrices by generalising a modified version of the three dimensional method presented in [7] to four dimensions.

The constraint equations (8) and (9) for $m = 2N$ and $m = 2N - 1$ are

$$\mathbf{A}_N^2 = \mathbf{B}_N^2,\ \ \mathbf{A}_N \cdot \mathbf{B}_N = 0 \tag{12}$$

and

$$\begin{aligned} &\mathbf{A}_N \cdot \mathbf{A}_{N-1} - \mathbf{B}_N \cdot \mathbf{B}_{N-1} = 0, \\ &\mathbf{A}_N \cdot \mathbf{B}_{N-1} + \mathbf{B}_N \cdot \mathbf{A}_{N-1} = 0 \end{aligned} \tag{13}$$

respectively. It follows from (12) that the highest harmonic is a circle that lives on some arbitrary plane. Clearly, we can introduce coordinates such that $\mathbf{A}_N$ and $\mathbf{B}_N$ lie on the w and x axes

$$\mathbf{A}_N = a\hat{w},\ \ \mathbf{B}_N = a\hat{x} \tag{14}$$

making the highest harmonic a circle of radius a on the w-x plane. This puts the vector $\tilde{\mathbf{b}}'_N$ into the so-called standard form [7]. Let $R_{wx}(\theta)$ be a matrix that rotates the w-x plane by an angle θ. Acting on $\tilde{\mathbf{b}}'_N$ with, $R_{wx}(-v)$ in these coordinates lowers the highest harmonic term of (7),

$$R_{wx}(-v) \begin{pmatrix} a\cos Nv \\ a\sin Nv \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a\cos(N-1)v \\ a\sin(N-1)v \\ 0 \\ 0 \end{pmatrix}. \tag{15}$$

This is not sufficient to verify that the overall harmonic content has been lowered because the $N-1$ terms of (7) could still give us an N harmonic through trigonometric identities. We now show, however, that this is not the case.

The constraint equations for $m = 2N - 1$ (13) give us using (14) the conditions on the coefficients,

$$(\mathbf{A}_{N-1})_w = (\mathbf{B}_{N-1})_x,\ \ (\mathbf{A}_{N-1})_x = -(\mathbf{B}_{N-1})_w. \tag{16}$$

It is not too difficult to see that using the conditions (16) when acting with $R_{wx}(-v)$ on the $N-1$ terms of (7) does not lead to the creation of N harmonic terms,

$$\begin{aligned} &R_{wx}(-v)\,[\mathbf{A}_{N-1}\cos(N-1)v + \mathbf{B}_{N-1}\sin(N-1)v] \\ &= \begin{pmatrix} (\mathbf{A}_{N-1})_w\cos(N-2)v - (\mathbf{A}_{N-1})_x\sin(N-2)v \\ (\mathbf{A}_{N-1})_x\cos(N-2)v + (\mathbf{A}_{N-1})_w\sin(N-2)v \\ (\mathbf{A}_{N-1})_y\cos(N-1)v + (\mathbf{B}_{N-1})_y\sin(N-1)v \\ (\mathbf{A}_{N-1})_z\cos(N-1)v + (\mathbf{B}_{N-1})_z\sin(N-1)v \end{pmatrix}. \end{aligned} \tag{17}$$

Clearly, the $N-2$ terms will not yield an N harmonic term when acted on by $R_{wx}(-v)$.

We now have a string in the form of (7) with $N \to N-1$. Its highest harmonic is therefore also a circle of some other radius living on some other arbitrary plane. Armed with this knowledge we can see that the unit magnitude constraints are such that in general we can write

$$\tilde{\mathbf{b}}'_N = R_{P_N}(v)\tilde{\mathbf{b}}'_{N-1} \tag{18}$$

where $R_{P_N}(v)$ is a rotation by an angle v on the plane P_N where the highest harmonic lives. By induction it must be that

$$\tilde{\mathbf{b}}'_N = \prod_{i=N}^{1} R_{P_i}(v)\tilde{\mathbf{b}}'_0 \tag{19}$$

where the $R_{P_i}(v)$ are rotations by an angle v on arbitrary planes P_i and $\tilde{\mathbf{b}}'_0$ is an arbitrary constant unit vector in 4 dimensions.

To specify a plane in 4 dimensions one needs to specify a direction on the plane (3 angles), a linearly independent direction (2 angles), but now the plane is overspecified by internal rotations (-1 angle), giving a total of 4 degrees of freedom for the matrices $R_{P_i}(v)$. For an N harmonic vector, one therefore has $4N$ parameters in the rotators and 3 parameters in the constant unit vector $\tilde{\mathbf{b}}'_0$ giving a total of $4N+3$ parameters which checks perfectly with $8N+4$ vector coefficients in (7) minus the $4N+1$ constraints (8) and (9).

The form of the $R_{P_i}(v)$ matrices is quite simple. Generally, if we want to transform, say, a rotation by an angle v on the w-x plane to one on an arbitrary plane we will need to perform a transformation of the type

$$R_{P_i}(v) = E_i R_{wx}(v) E_i^T. \tag{20}$$

For the purpose of finding the form of the rotators E_i it is easiest to envision the inverse process to the one we are seeking, namely the rotation of an arbitrary oriented plane to lie on the w-x plane. Let's consider first the projection of the four dimensional arbitrary plane onto the (x,y,z) subspace. We can perform a rotation by some angle α about the z axis ($R_{xy}(\alpha)$) until the vector perpendicular to the projected plane lies on the y-z plane and perform a further rotation by an angle β about the x axis ($R_{yz}(\beta)$) until that vector lies on the z axis. At this stage the projected plane lies wholly on the x-y plane. The ranges of both α and β from 0 to π are sufficient to perform these transformations. After performing these two rotations, our original four dimensional plane lies entirely in the (w,x,y) subspace and we can repeat an analogous process to the one above to rotate it into the w-x plane. A rotation by an angle γ about the y axis ($R_{wx}(\gamma)$) puts the vector perpendicular to the plane on the x-y plane and a rotation by an angle δ about w ($R_{xy}(\delta)$) makes that vector parallel with the y axis. For the first of these rotations a range of γ from 0 to π is sufficient, for the second rotation, however, matters are slightly different. If the plane we were trying to rotate was featureless it would be enough for the range of δ to be from 0 to π. In fact this is not the case. The plane contains a circle in v which can be oriented clockwise or anti-clockwise on the w-x plane and therefore the final rotation on the x-y plane in general requires an angle that ranges from 0 to 2π.

Keeping in mind these considerations we can quite generally write

$$E_i = R_{xy}(\alpha_i) R_{yz}(\beta_i) R_{wx}(\gamma_i) R_{xy}(\delta_i) \tag{21}$$

where α_i, β_i, γ_i range from 0 to π and δ_i ranges from 0 to 2π. Then

$$\tilde{\mathbf{b}}'_N(v) = \prod_{i=N}^{1} E_i R_{wx}(v) E_i^T \tilde{\mathbf{b}}'_0. \tag{22}$$

In order to construct the entire chiral string, we also need to find the form of $\mathbf{a}(u)$ in (2). The constraints (4) can be satisfied using a product of rotations that can be found from analogous arguments to the ones in the preceding section. For M harmonics this yields

$$\mathbf{a}'_M(u) = \prod_{i=M}^{1} D_i R^{(3)}_{xy}(u) D_i^T \mathbf{a}'_0 \tag{23}$$

with the rotator

$$D_i = R^{(3)}_{xy}(\phi_i) R^{(3)}_{yz}(\theta_i) \tag{24}$$

where the angles θ_i range from 0 to π, the angles ϕ_i from 0 to 2π and $R^{(3)}$ are the three dimensional rotation matrices.

III. OVERALL ORIENTATION FREEDOM

Both expressions for the oppositely moving excitations on the string (22) and (23) include overall orientation freedom. In some applications, for instance self-intersection or gravitational radiation analyses, only the shape of a loop, and not its orientation, is important. In this case the inclusion of overall orientation freedom of the right-moving and left-moving excitations separately is unnecessary: All that matters is the relative orientation between $\mathbf{a}'$ and the spatial part of $\tilde{\mathbf{b}}'$.

Overall orientation freedom of the loop is set by an Euler matrix Q that acts only on $\mathbf{a}'$ and the spatial components of $\tilde{\mathbf{b}}'$, which contains three angles. We consider the action of this matrix simultaneously on $\tilde{\mathbf{b}}'_N(v)$ and $\mathbf{a}'_M(u)$. The action of such a matrix on $\tilde{\mathbf{b}}'_N(v)$ can be verified to be

$$Q\tilde{\mathbf{b}}'_N(v) = \prod_{i=N}^{1} R_{P'_i}(v) Q\tilde{\mathbf{b}}'_0 \tag{25}$$

where

$$R_{P'_i}(v) = Q R_{P_i}(v) Q^T \tag{26}$$

with an analogous expression for $\mathbf{a}'_M(u)$. We are free to choose Q so that the first two angles eliminate the y and z components of $\tilde{\mathbf{b}}'_0(v)$,

$$Q\tilde{\mathbf{b}}'_0 = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \\ 0 \end{pmatrix} = R_{wx}(\alpha)\hat{w} \tag{27}$$

and the last angle acts on $\mathbf{a}'_0(u)$ to put it in the form

$$Q\mathbf{a}'_0 = \begin{pmatrix} \cos\beta \\ \sin\beta \\ 0 \end{pmatrix} = R_{xy}(\beta)\hat{x}. \tag{28}$$

It is important to note that the effect of replacing $R_{P_i}(v)$ with $R_{P_i'}(v)$ is to make the same solid body rotation on each of the planes of rotation, in other words, to rotate on some other set of planes. Since we can express a rotation on any plane using (21) however, the effect of the matrices Q on the rotators can be ignored.

Therefore, if we consider relative orientation freedom only, the forms of the right- and left-moving parts of the string now read

$$\tilde{\mathbf{b}}'_N(v) = \prod_{i=N}^{1} E_i R_{wx}(v) E_i^T R_{wx}(\alpha)\hat{w} \tag{29}$$

and

$$\mathbf{a}'_M(u) = \prod_{i=M}^{1} D_i R^{(3)}_{xy}(u) D_i^T R_{xy}(\beta)\hat{x}. \tag{30}$$

IV. THE CENTER OF MASS CONSTRAINT

In order to construct string loops, apart from solving the unit magnitude constraint, we need to satisfy the center of mass constraint, namely that the loop should be closed and that we want to work in the rest frame of the loop. These constraints imply that the center of mass term must vanish, $\mathbf{Z} = 0$ in (7), for both $\mathbf{a}$ and $\tilde{\mathbf{b}}$. In general this is an intractable problem because the center of mass term $\mathbf{Z}$ is a non-linear function of the angles in the rotation matrices. We can, however, solve this problem in an analogous way to that introduced in [10]. The guiding principle behind such a construction is to set the starting center of mass terms to zero, and to apply further rotation matrices in such a way as to ensure that trigonometric identities never lead to the production of a zero harmonic. In practice this was done by using odd harmonics only and choosing some parameters so that the starting center of mass term is zero. Here we will proceed along similar lines.

Generally, if the starting unit vector lies somewhere on the plane of the first rotation, $i = 1$ in (29) and (30), no center of mass term will be generated. Since our starting unit vectors are given by (27) and (28) we need to choose the first planes of rotations appropriately.

In three dimensions to rotate by an angle u on a plane that contains the x-axis, but is otherwise arbitrary, we can use

$$R_{yz}(\theta) R_{xy}(u) R_{yz}(-\theta). \tag{31}$$

If we now decide we want a rotation by u on a plane that contains a vector lying somewhere on the x-y plane, it is sufficient rotate this last matrix (31) on the x-y plane by whatever angle the vector makes with the x-axis. This is precisely the situation in (28). We therefore write the first rotation for (30) as

$$R^{(3)}_{xy}(\beta) R^{(3)}_{yz}(\theta_1) R^{(3)}_{xy}(u) R^{(3)}_{yz}(-\theta_1) R^{(3)}_{xy}(-\beta). \tag{32}$$

In four dimensions the situation is slightly more involved. We first consider a rotation by v on an arbitrary plane that contains the w-axis. Starting with $R_{wx}(v)$, we see that we can perform an arbitrary planar rotation on it in the (x, y, z) subspace that preserves the w-axis. This requires only two rotations. Therefore, to rotate about a plane that contains the w-axis but is otherwise arbitrary we can use

$$R_{xz}(\gamma) R_{xy}(\delta) R_{wx}(v) R_{xy}(-\delta) R_{yz}(-\gamma). \tag{33}$$

As before, we want to find the rotation v on a plane that contains a vector on the w-x plane because this is the situation of (27). It is not hard to see that the first rotator in (29) should be

$$R_{wx}(\alpha) R_{xz}(\gamma_1) R_{xy}(\delta_1) R_{wx}(v) R_{xy}(-\delta) R_{yz}(-\gamma) R_{wx}(-\alpha). \tag{34}$$

Further rotations should be by $2u$ and $2v$ to avoid the production of center of mass terms through trigonometric identities. These considerations yield

$$\tilde{\mathbf{b}}'_{2N-1}(v) = \prod_{i=N}^{2} E_i R_{wx}(2v) E_i^T \times R_{wx}(\alpha) R_{xz}(\gamma_1) R_{xy}(\delta_1) R_{wx}(v)\hat{w} \tag{35}$$

and

$$\mathbf{a}'_{2M-1}(u) = \prod_{i=M}^{2} D_i R^{(3)}_{xy}(2u) D_i^T \times R^{(3)}_{xy}(\beta) R^{(3)}_{yz}(\theta_1) R^{(3)}_{xy}(u)\hat{x}. \tag{36}$$

V. GENERALISATION TO HIGHER DIMENSIONS

Here we show the straightforward generalisation of our four dimensional argument in Section 2 to a unit vector living in arbitrary dimensions. The constraint equations (8) and (9) as well as the argument leading to (19) and (20) are independent of the number of dimensions the vector lives in. The only thing that changes with the number of dimensions is the parametrisation of E, the rotator that takes an arbitrary oriented plane to the plane of the first two coordinates. In the following, we label our spatial coordinates by the numbers 1 through d.

If we consider the projection of the plane onto the subspace given by the last three coordinates $(d-2, d-1, d)$ one can see that a rotation by α_d about the d axis $(R_{d-2,d-1}(\alpha_d))$ until the vector perpendicular to the projected plane lies in the d-$d{-}1$ plane followed by a rotation by an angle β_d about the $d-2$ axis $(R_{d-1,d}(\beta_d))$ is sufficient to rotate the projected plane out of the d axis.

To perform these transformations it is sufficient for the angles to range from 0 to π. We can repeat this procedure by moving up $d-2$ times in the coordinates until the plane lies entirely in the 1-2 plane as desired, ensuring that the range of the angle in the very last rotation $R_{2,3}(\beta_3)$ is from 0 to 2π to account for the fact that we are dealing with an oriented plane. We can then write the rotator as

$$E = R_{d-2,d-1}(\alpha_d) R_{d-1,d}(\beta_d) ... R_{1,2}(\alpha_3) R_{2,3}(\beta_3). \quad (37)$$

The number of parameters introduced by such a product is $2(d-2)$ per harmonic plus $(d-1)$ parameters to specify the initial unit vector giving a total of $2N(d-2)+d-1$ which checks with $d(2N+1)$ degrees of freedom in the vector coefficients of the Fourier series minus $4N+1$ constraints.

In the case of Nambu-Goto strings in an arbitrary number of dimensions d one would span both right and left moving excitations according to

$$\mathbf{a}'_N(u) = \prod_{i=N}^{1} E_i R_{1,2}(u) E_i^T \mathbf{a}'_0 \quad (38)$$

and

$$\mathbf{b}'_N(v) = \prod_{i=N}^{1} E_i R_{1,2}(v) E_i^T \mathbf{b}'_0 \quad (39)$$

with the E_i given by the a choice of (37) appropriate to the desired number of dimensions.

VI. CONCLUSIONS

We have generalised the solution to the unit magnitude constraint presented in [7] from three to four dimensions, casting it somewhat differently, in an effort to arrive at a general parametrisation of chiral superconducting strings with a finite number of harmonics. We have further shown how to construct loop solutions that satisfy the center of mass constraint and exclude overall orientation freedom. This result is useful because in studies of the properties of chiral loops, such as self-intersection and gravitational radiation properties, overall orientation of the loop is unimportant.

Studies of chiral cosmic string loops with constant currents [8] and simple varying currents [9] have been performed. Generally, however, we expect the current to be arbitrarily varying when loops are formed by intersections involving different strings or if different segments of the loop or string were at some point in causally disconnected regions. This is a fairly generic situation and a study of the properties of more general chiral loops should account for these variations. We hope that the technique presented here for solving the unit magnitude constraint in four dimensions will facilitate such a study.

Along the way, we have found that our modification of the method lends itself readily to a generalisation to arbitrary dimensions. We use such a generalisation to present solutions that could be useful in the investigation of classical relativistic strings in higher dimensions as well as strings in $3+1$ Minkowski space with currents and charges induced by Kaluza-Klein compactification [11] when the back-reaction from the gauge fields can be considered negligible.

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[1] T.W.B. Kibble, J. Phys. A**9**, 1387 (1976).
[2] A. Vilenkin and E.P.S Shellard, Cosmic strings and other Topological Defects. (Cambridge University Press, 1994).
[3] E. Witten, Nucl. Phys. B**249**, 557 (1985).
[4] B. Carter and P. Peter, Phys. Lett. B**466**, 41 (1999).
[5] J.J. Blanco-Pillado, K.D. Olum and A. Vilenkin, Phys. Rev. D**63**, 103513 (2001).
[6] K.D. Olum, J.J. Blanco-Pillado and X. Siemens, Nucl. Phys. B**599**, 446 (2001).
[7] R.W. Brown and D.B. DeLaney, Phys. Rev. Lett. **63**, 474 (1989); R.W. Brown, M.E. Convery and D.B DeLaney, J. Math. Phys. **32**, 1674 (1991).
[8] A.C. Davis, T.W.B Kibble, M. Pickles and D. Steer, Phys. Rev. D**62**, 083516 (2000).
[9] D.A. Steer, Phys. Rev. D**63**, 083517 (2001).
[10] X. Siemens and T.W.B Kibble, Nucl. Phys. B4**38**, 307 (1995).
[11] N.K. Nielsen, Nucl. Phys. B**167**, 249 (1980).